WEEK 3 DISCUSSION

FERMIONS & COMBINATORICS

SUPPOSE WE HAVE A SYSTEM OF N FERMIONS (E.G. ELECTRONS, PROTONS, ETC.) AND ANY SINGLE ~~EACH~~ FERMION CAN LIVE IN M DIFFERENT STATES.

IF THE FERMIONS WERE DISTINGUISHABLE WE COULD MAKE A BASIS TO DESCRIBE THEIR COLLECTIVE QUANTUM STATE USING THE VECTORS

$$v(n_1, \ldots, n_N)$$

WHERE THE n_j CAN TAKE VALUES $1, \ldots, M$

Q_1: WHAT IS THE DIMENSION OF THIS SPACE OF QUANTUM STATES? I.E. WHAT IS THE NUMBER OF BASIS VECTORS?

IDENTICAL FERMIONS, HOWEVER, ARE <u>NOT</u> DISTINGUISHABLE, AND THIS IS REFLECTED IN THE FACT THAT WE REQUIRE THAT ANY STATE BE ANTISYMMETRIC WHEN INTERCHANGE ANY TWO INDICES $n_i \longleftrightarrow n_j$, I.E. "TOTALLY ANTISYMMETRIC".

THAT IS, GIVEN ~~A~~ A STATE ψ, WHICH WE CAN ALWAYS WRITE IN THE FORM:

$$\psi = \sum_{n_1, \ldots, n_N = 1}^{M} \underbrace{C_{n_1, \ldots, n_N}}_{\text{complex numbers}} \underbrace{V(n_1, \ldots, n_N)}_{\text{basis vectors}}$$

WE REQUIRE THAT ψ SATISFY FOR ANY $i, j = 1, \ldots, N$

$$\pi_{ij} \psi = -\psi \quad , \text{ WHERE}$$

$$\pi_{ij} V(\ldots, n_i, \ldots, n_j, \ldots) = V(\ldots, n_j, \ldots, n_i, \ldots)$$

I.E. SWITCH n_i + n_j

A PERMUTATION ~~THE~~ π TAKES IN A NUMBER BETWEEN 1 AND N AND OUTPUTS ANOTHER NUMBER BETWEEN 1 AND N W/ THE RESTRICTION THAT NO TWO INPUTS HAVE THE SAME OUTPUT, i.e. A REARRANGEMENT OF THE NUMBERS $1, \ldots, N$.

Some properties of permutations:

- THERE ARE $N!$ OF THEM
- ACTING TWO PERMUTATIONS π, π' SUCCESSIVELY PRODUCES ANOTHER PERMUTATION.
- THE π_{ij} DEFINED EARLIER ARE PERMUTATIONS
- ANY PERMUTATION CAN BE CONSTRUCTED BY ~~ACT~~ STRINGING TOGETHER ENOUGH POSSIBLY DIFFERENT π_{ij}

Q2: IF WE DEFINE AN OPERATOR

π_A, THE ANTISYMMETRIZING OPERATOR,

BY THE RULE

~~$\pi_A v(\ldots, n_i, \ldots)$~~ $\pi_A =$

$$= \sum_{\pi} (-1)^{\sigma(\pi)} \pi = \sum_{\pi} \sigma(\pi)\, \pi$$

WHERE THE SUM IS OVER ALL

PERMUTATIONS π AND $\sigma(\pi)$ IS

+1 IF π IS CONSTRUCTED FROM

AN EVEN ~~ODD~~ NUMBERED STRING OF π_{ij},

AND −1 IF ODD, THEN SHOW THAT

FOR ANY STATE ψ THE STATE

$\pi_A \psi$ IS ANTISYMMETRIC WRT

ANY OF THE π_{ij}.

HINT: PROVE IT FOR THE $v(\ldots, n_i, \ldots)$ FIRST.

Q_2 CONT

FURTHER
HINT: FOR ANY function OF THE PERMUTATIONS $f(\pi)$, WE HAVE THE FOLLOWING:

$$\sum_{\pi} f(\pi'\pi) = \sum_{\pi} f(\pi)$$

WHERE π' IS AN ARBITRARY PERMUTATION.

ASIDE:
THE ABOVE TRICK UNDERLIES MUCH OF THE APPLICABILITY OF GROUP THEORY TO PHYSICAL PROBLEMS [THE PERMUTATIONS FORM A GROUP] ~~THE~~

Q3 : SHOW THAT ANY ~~ANTIS~~ TOTALLY ANTISYMMETRIC STATE

LIES IN THE IMAGE OF Π_A, SO

THAT IT CAN BE EQUATED W/ THE

SPACE OF VALID FERMION STATES.

HINT: SUPPOSE YOU HAVE A

TOTALLY ANTISYMMETRIC STATE

ψ AND OPERATE ON IT W/ Π_A.

WHAT DO YOU FIND? CAN YOU

FIND THEN ANOTHER STATE ψ' SO

THAT $\Pi_A \psi' = \psi$?

Q4: SHOW THAT ANY BASIS VECTOR

$v(\ldots, n_i, \ldots, n_j, \ldots)$ WITH

$n_i = n_j$ IS "ANNIHILATED" BY π_A, i.e.

$$\pi_A v(\ldots, n_i, \ldots, n_j, \ldots) = 0$$

~~HINT~~: USE (OR PROVE) THE FACT THAT

$$[\pi_A, \pi_{ij}] \equiv \pi_A \pi_{ij} - \pi_{ij} \pi_A = 0$$

$$\text{i.e. } \pi_A \pi_{ij} = \pi_{ij} \pi_A$$

$$\text{FOR ANY } i, j = 1, \ldots, N$$

Q5: SHOW THAT IF TWO BASIS VECTORS

$v \equiv v(\ldots, n_i, \ldots)$ AND $v' \equiv v(\ldots, n'_i, \ldots)$

~~$v \;\; v$~~ ARE RELATED BY A

PERMUTATION π, i.e.

$$\pi v(\ldots, n_i, \ldots) = v(\ldots, n'_i, \ldots)$$

~~THEN A THEN $\pi_A(v)$ AND π_A~~

THEN $\pi_A v$ AND $\pi_A v'$ ARE LINEARLY DEPENDENT. THEN PROVE THE CONVERSE, THAT IF BASIS VECTORS v, v' ARE LINEARLY DEPENDENT THEN THEY ARE RELATED BY A PERMUTATION $v' = \pi v$

~~Q~~: So we have established ~~two~~ three main points:

- The image of π_A is the space of fermion states
- Basis vectors w/ ~~repeating~~ duplicate indices, i.e. $v(\ldots, n_i, \ldots, n_j, \ldots)$ where $n_i = n_j$, are annihilated by π_A.
- Basis vectors ~~w/~~ that are related by permutations have the same image, and have different images if they do not.

Q6: What then is the dimension of the space of fermions in terms of N and M?

ANSWERS

A₁: EACH INDEX $n_1, \ldots, n_N$ CAN TAKE ONE OF M VALUES SO THERE ARE $\underbrace{M \times \cdots \times M}_{N} = M^N$ DIFFERENT BASIS VECTORS

A₂: $\pi_{ij}\pi_A =$

$$\pi_{ij}\sum_{\pi} \sigma(\pi)\,\pi$$

$$= \sum_{\pi} \sigma(\pi)\,\pi_{ij}\pi$$

$$= \cancel{\sum_{\pi}} \sum_{\pi} (-1)^2 \sigma(\pi)\,\pi_{ij}\pi$$

$$= (-1) \sum_{\pi} \sigma(\pi_{ij})\,\sigma(\pi)\,\pi_{ij}\pi$$

$$= (-1) \sum_{\pi} \sigma(\pi_{ij}\pi)\,\pi_{ij}\pi$$

(!) $$= (-1) \sum_{\pi} \sigma(\pi)\,\pi$$

$$= -\pi_A$$

SO FOR ANY ψ:

$$\pi_{ij}\left[\pi_A \psi\right] = -\pi_A \psi$$

I MADE USE OF THE FACT THAT

ACTING TWO PERMUTATIONS AS OPERATORS IN SEQUENCE IS THE SAME AS ACTING THEIR COMBINED PERMUTATION AS A SINGLE OPERATOR. WE CHECK THAT THIS IS SO USING THE BASIS VECTORS:

$$\pi' \pi \, v(\ldots, n_i, \ldots)$$

$$= \pi' \, v(\ldots, n_{\pi(i)}, \ldots)$$

$$= v(\ldots, n_{\pi'(\pi(i))}, \ldots)$$

$$= v(\ldots, n_{\pi' \circ \pi(i)}, \ldots)$$

$$= (\pi' \circ \pi) \, v(\ldots, n_i, \ldots)$$

WHERE THE "COMPOSITION" OPERATOR $\circ$ MEANS TO ACT THEM IN SEQUENCE.

$\underline{A_3}$: SUPPOSE WE HAVE A ψ W/

~~π_A~~ $\pi_{ij} \psi = -\psi$ FOR ANY i, j.

WE NEED TO SHOW THAT THERE

IS SOME STATE ψ' W/ $\pi_A \psi' = \psi$.

LOOKING @ $\pi_A \psi$ WE FIND:

$$\pi_A \psi = \sum_{\pi} \sigma(\pi)\, \pi\, \psi$$

WHY?

(!) $$= \sum_{\pi} \sigma(\pi)\, \sigma(\pi)\, \psi$$

$$= \sum_{\pi} \psi$$

$$= N!\, \psi$$

SO $\pi_A \frac{1}{N!} \psi = \psi$ ✓

A_4: FIRST WE SHOW THAT

$\pi_A \pi_{ij} = \pi_{ij} \pi_A$ FOR ANY i, j:

$\pi_A \pi_{ij} = \cancel{\sum_{\pi} \sigma(\pi) \pi \pi_{ij}}$ $\sum_{\pi} \sigma(\pi) \pi \pi_{ij}$

$= \pi_{ij}^2 \sum_{\pi} \sigma(\pi) \pi \pi_{ij}$

$= \pi_{ij} \sum_{\pi} \sigma(\pi_{ij} \pi \pi_{ij}) \pi_{ij} \pi \pi_{ij}$

$= \pi_{ij} \sum_{\pi} \sigma(\pi) \pi$

$= \pi_{ij} \pi_A$ ✓

SO IF WE HAVE A ~~S~~ BASIS

VECTOR v W/ $\pi_{ij} v = v$ THEN

$\pi_A v = \pi_A \pi_{ij} v = \pi_{ij} \pi_A v$

$= -\pi_A v \Rightarrow \pi_A v = 0$ ✓

A5: $\pi_A v' = \pi_A \pi v = \pi \pi_A v = \pm \pi_A v$ ✓

$= \cancel{\sum_{\pi'} \sigma(\pi') \pi \pi' v =}$

A6: WAYS OF PICKING N STATES OUT OF M POSSIBLE!

$$\mathrm{Dim}(\pi_A) = \binom{M}{N}$$

$$= \frac{M!}{(M-N)!\,N!}$$