ANSWERS

1Ai OBJECTS IN MOTION TEND TO STAY IN MOTION!

$$\vec{x}(t) = \vec{x}_0 + \vec{v}\,t$$

1Aii JUST NEWTON'S LAWS AGAIN. THE FORCE IS EXERTED FOR SUCH A SHORT TIME ~~THE~~ THAT THE FORCE IS ESSENTIALLY CONSTANT.

$$\frac{d\vec{p}(t)}{dt} = \vec{F}(\vec{x}(t)) \quad \text{OR} \quad \frac{d\vec{v}(t)}{dt} = \frac{\vec{F}(\vec{x}(t))}{m}$$

~~$\Delta\vec{p}$~~

$$\Delta\vec{v} = \vec{v}(t_0+\Delta t) - \vec{v}(t_0)$$

$$= \int_{t_0}^{t_0+\Delta t} dt\, \frac{\vec{F}(\vec{x}(t))}{m}$$

$$\simeq \int_{t_0}^{t_0+\Delta t} dt\, \frac{\vec{F}(\vec{x}_0)}{m}$$

$$= \frac{\vec{F}(\vec{x}_0)}{m} \int_{t_0}^{t_0+\Delta t} dt$$

$$= \frac{\vec{F}(\vec{x}_0)}{m}\left[t_0+\Delta t - t_0\right] = \Delta t\,\frac{\vec{F}(\vec{x}_0)}{m}$$

1Aiii $\Delta \vec{v} = \vec{v}_{FINAL} - \vec{v}_{INITIAL}$

$$= \int_{-\infty}^{\infty} dt \, \frac{\vec{F}(\vec{x}(t))}{m}$$

!!! $$\simeq \int_{-\infty}^{\infty} dt \, \frac{\vec{F}(\vec{x}_0 + \vec{v}t)}{m}$$

WHERE $\vec{F}(\vec{x}) = -\vec{\nabla} V(\vec{x})$

SO WHILE WE ADMIT THE VELOCITY IS CHANGING, WE ASSERT THAT IT DOES NOT CHANGE MUCH SO THAT THE FORCE EXERTED ON THE PARTICLE IS APPROXIMATELY ~~UNCHANGED~~ IDENTICAL TO THE FORCE EXPERIENCED BY THE UNIFORM MOTION TRAJECTORY.

1Aiv FOR CENTRAL POTENTIALS (i.e. DEPENDING ONLY ON DISTANCE BETWEEN SCATTERERS) MOTION IS CONFINED TO A PLANE, SO WE ONLY NEED TO CONSIDER MOTION IN PLANE OF $\vec{v}$ AND $\vec{b}$ (SEE DIAGRAM). THE ELECTRON'S PRIMARY TRAJECTORY CAN BE WRITTEN:

$$\vec{x}(t) = b\hat{y} + vt\hat{x}$$

AND THE FORCE OF A COULOMB INTERACTION IS

$$\vec{F}(t) = k(-e)(Ze)/r^2 \cdot \hat{r}$$

USING THE RESULT FROM 1Aiii WE COMPUTE VELOCITY CHANGE:

$$\Delta\vec{v} = \int_{-\infty}^{\infty} dt \, \frac{\vec{F}(b\hat{y} + vt\hat{x})}{m}$$

$$= \frac{-kZe^2}{m} \int_{-\infty}^{\infty} dt \, \frac{\hat{r}(t)}{(b^2 + (vt)^2)}$$

$$\hat{r}(t) = \frac{\vec{r}(t)}{r} = \frac{x(t)\hat{x} + y(t)\hat{y}}{r}$$

$$= \frac{-kZe^2}{m}\left[\int_{-\infty}^{\infty} dt \, \frac{vt\hat{x}}{[b^2 + (vt)^2]^{3/2}} + \int_{-\infty}^{\infty} dt \, \frac{b\hat{y}}{[b^2 + (vt)^2]^{3/2}}\right]$$

THE FIRST INTEGRAL IS ZERO SINCE THE NUCLEUS PULLS THE ELECTRON TO THE LEFT ON ITS WAY OUT JUST AS HARD AS IT PULLS IT RIGHT ON ITS WAY IN.

$$= \frac{-kZe^2}{m} \int_{-\infty}^{\infty} dt \, \frac{\hat{y}\, b}{b^3 \left[1 + \left(\frac{vt}{b}\right)^2\right]^{3/2}}$$

$$= \frac{-kZe^2}{mvb} \int_{-\infty}^{\infty} dz \cdot \frac{\hat{y}}{\left[1 + z^2\right]^{3/2}}$$

$$= \frac{-kZe^2}{mvb} \cdot 2 \hat{y}$$

$$= -v \cdot \left(\frac{kZe^2}{b} \Big/ \frac{1}{2} m v^2 \right)$$

SO THE ELECTRON IS DEFLECTED BY THE RATIO OF THE BINDING ENERGY AT CLOSEST APPROACH AND THE KINETIC ENERGY. THE VALIDITY OF THE APPROXIMATION REQUIRES THAT THIS RATIO BE SMALL.

$$\theta = \text{ARCTAN}\left(\frac{V_y}{V_x}\right)$$

$$= \text{ARCTAN}\left(\frac{-v\left[\frac{kZe^2}{b} \Big/ \frac{1}{2} m v^2\right]}{v} \right)$$

$$\simeq -\frac{kZe^2}{b} \Big/ \frac{1}{2} m v^2$$

1Bi $\psi(\vec{x}+\Delta\vec{x}, t) =$

$$\psi_0 \exp\left[i\left[\omega t - \frac{\omega}{c}\left|\vec{x}+\Delta\vec{x}-\vec{x}_0\right|\right]\right]$$

$$\left|\vec{x}-\vec{x}_0+\Delta\vec{x}\right|$$

WE ASSUME THE "SPREADING LOSS" $\dfrac{1}{|\vec{x}+\Delta\vec{x}-\vec{x}_0|}$ IS ESSENTIALLY CONSTANT IN THE VICINITY OF $\delta\vec{x}$ & IS EQUAL TO

$$1/|\vec{x}-\vec{x}_0| \equiv 1/r$$

LOOKING THEN AT THE ARGUMENT OF THE EXPONENT :

$$\xi \quad \left|\vec{x}+\Delta\vec{x}-\vec{x}_0\right|$$

$$= \left[(\vec{x}+\Delta\vec{x}-\vec{x}_0)\cdot(\vec{x}+\Delta\vec{x}-\vec{x}_0)\right]^{1/2}$$

$$\vec{r} \equiv \vec{x}-\vec{x}_0$$

$$= \left[(\vec{r}+\Delta\vec{x})\cdot(\vec{r}+\Delta\vec{x})\right]^{1/2}$$

$$= \left[r^2 + 2\,\vec{r}\cdot\Delta\vec{x} + \cancel{\Delta x^2}\right]^{1/2} \quad \cancel{\Delta x^2} \to 0$$

(MUCH SMALLER THAN $\vec{r}\cdot\Delta\vec{x}$)

$$\simeq r\left[1+\frac{1}{2}\cdot 2\,\frac{\vec{r}\cdot\Delta\vec{x}}{r^2}\right]$$ TAYLOR EXPANSION

$$= r + \hat{r} \cdot \Delta\vec{x}$$

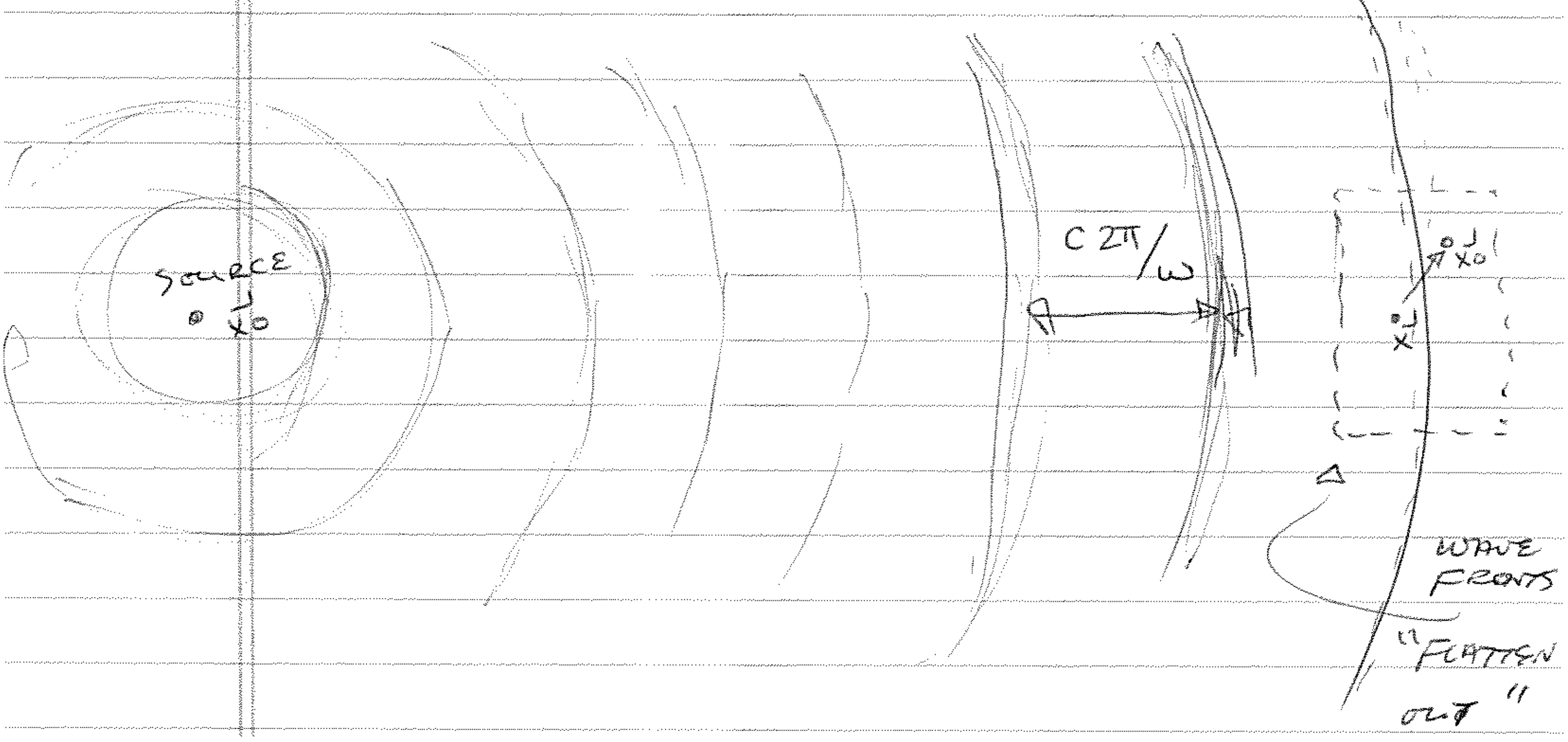
plugging this in:

$$\psi(\Delta\vec{x} + \vec{x}, t) \simeq$$

$$\frac{\psi_0}{r} \exp\left[i\left[\omega t - \frac{\omega}{c}\left[r + \hat{r}\cdot\Delta\vec{x} \right] \right] \right]$$

$$= \left[\frac{\psi_0}{r} \exp\left[i \frac{\omega}{c} r \right] \right] \exp\left[i\left[\omega t - \left[\frac{\omega}{c}\hat{r} \right]\cdot\Delta\vec{x} \right] \right]$$

So in the vicinity of $\vec{x}$, the wave from the source @ $\vec{x}_0$ looks like a plane wave moving in direction $\hat{r} \propto \vec{x} - \vec{x}_0$ w/ wavelength $c/\omega/2\pi = \frac{c}{f}$.

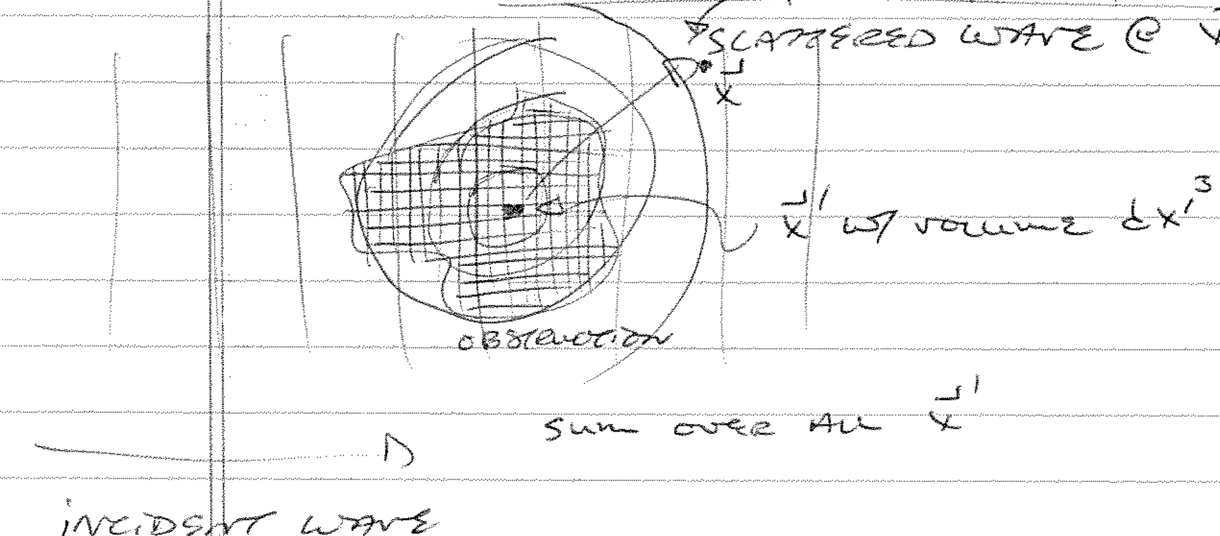


1B ii) IN THE BORN APPROX WE IGNORE MULTIPLE SCATTERINGS SO WE JUST ADD UP ALL THE LITTLE POINT OBSTRUCTIONS GENERATING SPHERICAL WAVES:

$$\phi(\vec{x},t) = \int \Delta\phi(\vec{x},t;\vec{x}')$$

$$= \int dx'^3 \, A(\vec{x}') \, \frac{\psi\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}$$

SCATTERED WAVE @ $\vec{x}$



1Biii ~~A~~ SINCE THE POINT SOURCE IS FAR AWAY FROM THE ARRAY, IT IS ESSENTIALLY A PLANE WAVE:

$$\psi(\vec{x}',t') \sim \exp\left(i\left[\omega t' - \frac{\omega}{c}\hat{r}'\cdot\vec{x}'\right]\right)$$

[NOT INTERESTED IN SPREADING FACTOR $\frac{1}{r}$ OR SOURCE AMP. SINCE THEY ARE CONSTANT OVER ARRAY]

SO $\phi(\vec{x},t) =$

$$\int dx'^3 \frac{A(\vec{x}')}{\cancel{|\vec{x}-\vec{x}'|}} \psi\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

D ESSENTIALLY CONSTANT SINCE $\vec{x}$ ~~MUCH~~ ~~FURTHER~~ VERY FAR AWAY.

$$\sim \int dx'^3 \sum_{lmn=0}^{N} \delta(x'-la)\,\delta(y'-ma)\,\delta(z'-na)$$

$$\cdot \exp\left(i\left[\omega\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right) - \frac{\omega}{c}\hat{r}'\cdot\vec{x}'\right]\right)$$

FOR SAME REASON AS 1Bi WE CAN USE LARGENESS OF $\vec{x}$ TO SIMPLIFY ~~TERM~~ $|\vec{x}-\vec{x}'|$ TO: $x - \frac{1}{2}\frac{2\,\vec{x}\cdot\vec{x}'}{x}$

$$\sim \sum_{lmn=0}^{N} \int dx'^3 \, \delta(x'-la)\,\delta(y'-ma) \cdot \delta(z'-na) \cdot \mathrm{Exp}\left(i\omega\left[t - \frac{[\hat{r}'-\hat{r}]\cdot \vec{x}'}{c}\right]\right)$$

(IGNORING PHASE FACTORS)

DEFINING $-\vec{q} \equiv \frac{\omega}{c}[\hat{r}'-\hat{r}]$

+ ~~ALL~~ INTEGRATING:

$$= \sum_{lmn=0}^{N} \mathrm{Exp}\left[i\omega t + \vec{q}\cdot a\left[l\hat{x} + m\hat{y} + n\hat{z}\right]\right]$$

IGNORING PHASES

$$\sim \cancel{\mathrm{Exp}[i\omega t]}$$

$$\sum_{l}^{N} \mathrm{Exp}[i q_x a \cdot l] \sum_{m}^{N} \mathrm{Exp}(i q_y a \cdot m] \cdot \sum_{n}^{N} \mathrm{Exp}(i q_z a \cdot n)$$

SINCE $\mathrm{Exp}[i q_x a\, l] = \left(\mathrm{Exp}[i q_x a]\right)^l$
WE CAN RECOGNIZE THIS AS PRODUCT OF 3 PARTIAL GEOMETRIC SERIES, WHICH AFTER SOME REARRANGING AND DISCARDING PHASE FACTORS, BECOMES:

$$\sim \frac{\sin((N+1) q_x a)}{\sin(q_x a)} \cdot \frac{\sin((N+1) q_y a)}{\sin(q_y a)} \cdot \frac{\sin((N+1) q_z a)}{\sin(q_z a)}$$

FOR ~~LARGER~~ LARGE N
~~THESE~~ THIS ~~FULL~~ EXPRESSION IS SHARPLY PEAKED AT

$$q_x a = n\pi,$$
$$q_y a = m\pi \qquad (A)$$
$$q_z a = l\pi$$

FOR SOME INTEGERS n, m, l

~~SO THAT~~ BUT $\vec{q}$ WAS DEFINED AS:

$$\frac{\omega}{c}\left[\hat{r}' - \hat{r}\right]$$

SINCE $\hat{r}$ IS DEFINED BY TWO ANGLES θ, ϕ

AND SO ONLY TAKES VALUES ALONG SOME 2-DIMENSIONAL SURFACE OF ALLOWABLE $\vec{q}$ VECTORS (THIS IS ESSENTIALLY ENERGY CONSERVATION), WHILE THE CUBIC ARRAY ~~ALL~~ ONLY SCATTERS WHEN $\vec{q}$ HAPPENS TO BE ONE OF SOME DISCRETE SET OF POINTS ~~ANGLES~~ SATISFYING (A). IT WOULD BE A COINCIDENCE FOR ANY OF THESE POINTS TO HAPPEN TO LIE ON THE "ALLOWED SURFACE" OF $\vec{q}$-VALUES, AND IN GENERAL THEY DO NOT. THIS IS WHY CRYSTALS ARE TRANSPARENT.

1C NON-LINEAR LIGHT PROPAGATION, LINEARIZATION OF FLUID MECHANICAL EQUATIONS, ...

2A JUST SOME ALGEBRA:

$$\frac{p^2}{2m} + \frac{1}{2} m \cdot x^2 \cdot \omega^2$$

$$= \frac{\hbar\omega}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2\right)$$

$$= \hbar\omega\left(\frac{p^2}{2m\hbar\omega} + \frac{x^2 m\omega}{2\hbar}\right)$$

$$= \frac{\hbar\omega}{2}\left(\left[p/\sqrt{m\hbar\omega}\right]^2 + \left(x/\sqrt{\hbar/m\omega}\right)^2\right)$$

$$= \hbar\omega \cdot \frac{1}{2}\left[\left(p/p_0\right)^2 + \left(x/x_0\right)^2\right]$$

$$= \hbar\omega \cdot \frac{1}{2}\left[\tilde{p}^2 + \tilde{x}^2\right]$$

DIVIDING OUT BY $\hbar\omega$ TO PUT HAMILTONIAN IN UNITS OF $\hbar\omega$:

$$\tilde{H}_0 = \frac{1}{2}\left(\tilde{p}^2 + \tilde{x}^2\right)$$

$$p_0 = \left[m\hbar\omega\right]^{1/2} \qquad x_0 = \left[\hbar/m\omega\right]^{1/2}$$

(2B) $H' = \alpha x^4$

$= \hbar\omega \dfrac{\alpha}{\hbar\omega}\left(x_0 \tilde{x}\right)^4$

$\tilde{H}' = \dfrac{\alpha}{\hbar\omega} x_0^4 \tilde{x}^4$

$= \beta \tilde{x}^4$

$\beta \equiv \dfrac{\alpha x_0^4}{\hbar\omega} \qquad \tilde{H}' \equiv \dfrac{H'}{\hbar\omega}$

β IS UNITLESS. CAN SEE THIS TWO WAYS:

- $\tilde{H}'$ IS UNITLESS & SO IS $\tilde{x}$, SO β MUST BE AS WELL.

- H' IS ENERGY & $H' = \alpha x^4$
  SO α IS ~~LENGTH~~ $\dfrac{\text{ENERGY}}{\text{LENGTH}^4}$

SO $\dfrac{\alpha x_0^4}{\hbar\omega} = \dfrac{\cancel{\text{ENERGY}}/\cancel{\text{LENGTH}^4} \cdot \cancel{\text{LENGTH}^4}}{\cancel{\text{ENERGY}}}$

$=$ UNITLESS.

CHECK THIS!

2C

$$a_\pm = \frac{1}{\sqrt{2}}\left[\frac{x}{x_0} \mp i \frac{p}{p_0}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\tilde{x} \mp i \tilde{p}\right]$$

so $$\tilde{x} = \frac{1}{\sqrt{2}}\left[a_+ + a_-\right]$$

$$H' = \frac{\beta}{4}\left[a_+ + a_-\right]^4$$

~~$= \beta/4$~~
~~++++ + +++− + ++−+ + ++−− +~~
~~+−++ + +−+− + +−−+ + +−−− +~~
~~−+++ + −++− + −+−+ + −−+− +~~
~~−−++ + −−+− +~~

DEFINING $a_+ \equiv 1$ $a_- \equiv 0$

WE EXPAND AS:

$$= \beta/4 \left(\begin{array}{llll} 1111 + & 1110 + & 1101 & + 1100 \\ 1011 + & 1010 & + 1001 & + 1000 \\ 0111 + & 0110 & + 0101 & + 0100 \\ 0011 + & 0010 & + 0001 & + 0000 \end{array} \right)$$

2D • ~~CAN'T HAVE~~ MUST HAVE SAME NUMBER OF a_+ & a_-, OTHERWISE OPERATING ALL FOUR ON KET WILL GIVE STATE ORTHOGONAL TO $\langle 0|$

- CAN'T END IN a_-, OTHERWISE ANNIHILATES $|0\rangle$

- CAN'T BEGIN W/ a_+ SINCE:

$$\langle 0 | a_+$$

$$= \left[a_+^\dagger |0\rangle \right]^\dagger$$

$$= \left[a_- |0\rangle \right]^\dagger$$

$$= \left[0 \right]^\dagger$$

$$= 0$$

APPLYING THESE RULES, ONLY LEFT W/:

$a_- a_- a_+ a_+$, $a_- a_+ a_- a_+$ ~~$a_- a_+ a_+ a$~~

2E

$$\langle 0 | a_- a_- a_+ a_+ | 0 \rangle$$

$$= \| a_+ a_+ |0\rangle \|^2 \qquad a_-^\dagger = a_+$$

$$= \| a_+ \sqrt{1}\, |1\rangle \|^2$$

$$= \| \sqrt{2}\sqrt{1}\, |2\rangle \|^2$$

$$= 2 \langle 2 | 2 \rangle = 2$$

$\langle 0 | a_- a_+ a_- a_+ | 0 \rangle$

$= \| a_- a_+ |0\rangle \|^2$

$= \| \sqrt{1}\sqrt{1} \, |0\rangle \|^2$

$= \langle 0 | 0 \rangle = 1$

So $E_0' = \langle 0 | H' | 0 \rangle$

$= \hbar\omega \cdot \frac{\beta}{4} \left[2 + 1 \right]$

$= \frac{3}{4} \hbar\omega \cdot \beta$

CORRECTION IS SMALL IF:

$\beta \ll 1$

OR $\frac{\alpha x_0^4}{\hbar\omega} \ll 1$

OR $\alpha \ll \frac{\hbar\omega}{x_0^4}$