WEEK 4

ANSWERS

A1 $$H\left[A|\psi\rangle\right] = \left[HA\right]|\psi\rangle$$

$$= \left[AH\right]|\psi\rangle = A\left[H|\psi\rangle\right]$$

$$= A\left[\lambda|\psi\rangle\right] = \lambda\left[A|\psi\rangle\right]$$

So $$H\left[A|\psi\rangle\right] = \lambda\left[A|\psi\rangle\right]$$

AND THUS $A|\psi\rangle$ IS AN EIGENSTATE OF H WITH EIGENVALUE λ

A2 $$\lambda\langle\psi'|A\psi\rangle = \langle\psi'|AH\psi\rangle$$

$$= \langle\psi'|HA\psi\rangle = \langle H\psi'|A\psi\rangle$$

$$= \lambda'\langle\psi'|A\psi\rangle$$

So $$\lambda\langle\psi'|A\psi\rangle = \lambda'\langle\psi'|A\psi\rangle$$

OR: $$(\lambda' - \lambda)\langle\psi'|A\psi\rangle = 0$$

SINCE $\lambda' \neq \lambda$ THIS MEANS THAT

$$\langle\psi'|A\psi\rangle = 0$$

A3 AS THE HINT SUGGESTS, TAKE ANY ~~BASI~~ EIGENBASIS OF H:

$$\left(|\psi_{11}\rangle, |\psi_{12}\rangle, \ldots, |\psi_{ij}\rangle, \ldots \right)$$

WHERE $H|\psi_{ij}\rangle = \lambda_i |\psi_{ij}\rangle$

FROM Q2 WE KNOW THAT:

$$\langle \psi_{ab} | A \psi_{ij} \rangle = 0 \quad \text{IF} \quad a \neq i$$

SO WE CAN FOR EACH ENERGY LEVEL (EACH i) CONSTRUCT THE MATRIX

$$A^i_{jk} = \langle \psi_{ij} | A \psi_{ik} \rangle$$

AND SOLVE FOR THE EIGENVALUES λ'^i_j AND EIGENVECTORS $\vec{a}^i_j$ SO THAT:

$$A^i \vec{a}^i_j = \lambda'^i_j \vec{a}^i_j \quad \text{OR:}$$

$$\cancel{\sum_{kl} A} \quad \sum_l A^i_{kl} a^i_{jl} = \lambda'^i_j a^i_{jk}$$

WHERE $\vec{a}^i_j = \begin{bmatrix} a^i_{j1} \\ a^i_{j2} \\ \vdots \\ a^i_{jk} \\ \vdots \end{bmatrix}$

WE THEN DEFINE A NEW BASIS:

$$\left(|\psi'_{11}\rangle, |\psi'_{12}\rangle, \ldots, |\psi'_{ij}\rangle, \ldots \right)$$

WHERE:

$$|\psi'_{ij}\rangle = \sum_k a^i_{jk} |\psi_{ik}\rangle$$

AND CHECK THAT:

$$H|\psi'_{ij}\rangle = H\sum_k a^i_{jk} |\psi_{ik}\rangle$$

$$= \sum_k a^i_{jk} H|\psi_{ik}\rangle$$

$$= \sum_k a^i_{jk} \lambda_i |\psi_{ik}\rangle$$

$$= \lambda_i \sum_k a^i_{jk} |\psi_{ik}\rangle$$

$$= \lambda_i |\psi'_{ij}\rangle$$

(i.e. THE NEW VECTORS ARE JUST SUMS OF DEGENERATE VECTORS)

AND, MOREOVER:

$$A\,|\psi'_{ij}\rangle$$

$$= A \sum_k a^i_{jk}\,|\psi_{ik}\rangle$$

$$= \sum_{lm} |\psi_{lm}\rangle\langle\psi_{lm}|\,A \sum_k a^i_{jk}\,|\psi_{ik}\rangle$$

$$= \sum_{klm} a^i_{jk}\,|\psi_{lm}\rangle\langle\psi_{lm}|\,A\,\psi_{ik}\rangle$$

$$= \sum_{klm} a^i_{jk}\,|\psi_{lm}\rangle\,\delta_{li}\,\langle\psi_{lm}|\,A\,\psi_{ik}\rangle$$

$$= \sum_{km} a^i_{jk}\,|\psi_{im}\rangle\,A^i_{mk}$$

$$= \sum_m |\psi_{im}\rangle \sum_k A^i_{mk}\,a^i_{jk}$$

$$= \sum_m |\psi_{im}\rangle\,\lambda'^i_j\,a^i_{jm}$$

$$= \lambda'^i_j \sum_m a^i_{jm}\,|\psi_{im}\rangle$$

$$= \lambda'^i_j\,|\psi'_{ij}\rangle$$

SO NOW THE NEW BASIS IS ALSO AN EIGENBASIS OF A.

(A4) IF THERE IS ONLY ONE EIGENBASIS OF A, AND (Q3) GAURANTEES AT LEAST ONE BASIS DIAGONALIZING BOTH A & H, THEN THIS BASIS MUST DIAGONALIZE H AS WELL.

(A5) LET $|\psi\rangle$ AND $|\phi\rangle$ BE VECTORS FROM Λ, i.e. $H|\psi\rangle = \lambda|\psi\rangle$

AND $H|\phi\rangle = \lambda|\phi\rangle$

AND LET c BE A SCALAR.

THEN:

$$H\left[|\psi\rangle + c|\phi\rangle\right] = H|\psi\rangle + cH|\phi\rangle$$

$$= \lambda|\psi\rangle + c\lambda|\phi\rangle$$

$$= \lambda\left[|\psi\rangle + c|\phi\rangle\right]$$

So ~~THAT~~ THUS $|\psi\rangle + c|\phi\rangle$ IS ALSO AN EIGENSTATE OF H WITH EIGENVALUE λ AND SO IS IN Λ.

(A6) SINCE FROM (Q1) WE KNOW THAT IF $|\psi\rangle$ IS IN Λ THEN $A|\psi\rangle$ IS ALSO IN LAMBDA, THEN WE CAN PROJECT OUT ANY PERPENDICULAR ~~COMPONENTS~~ COMPONENTS OF ~~BY APPLY~~ $A i_\Lambda |\psi\rangle$ WITHOUT CHANGING ANYTHING SINCE $A i_\Lambda |\psi\rangle$ IS GAURANTEED TO LIE IN Λ. IN OTHER WORDS, WE CAN

TACK ON A $i_\wedge P_\wedge$ TO THE END OF $A i_\wedge$ WITHOUT CHANGING ANYTHING. THEN:

$$A i_\wedge = \left[i_\wedge P_\wedge \right] A i_\wedge = i_\wedge \left[P_\wedge A i_\wedge \right]$$

$$= i_\wedge A|_\wedge$$

SIMILARLY:

$$P_S A = P_S A \left[i_S P_S \right] = \left[P_S A i_S \right] P_S$$

$$= A|_S P_S$$

A7 $$A|_S H'|_S = A|_S P_S H' i_S$$

$$= P_S A H' i_S$$

$$= P_S H' A i_S$$

$$= P_S H' i_S A|_S$$

$$= H'|_S A|_S$$

THEREFORE $\left[A|_S , H'|_S \right] = 0$

AND SO $A|_S$ IS A SYMMETRY OF $H'|_S$

(A8) YOU SEE IT OR YOU DON'T. YOU CAN REFLECT THE POTENTIAL ACROSS THE LINE $x=y$ AND GET THE SAME POTENTIAL. THIS IS THE SAME AS SENDING EVERY POINT (x,y) TO THE POINT (y,x)

(A9) A) $\left(A\left[\psi + c\phi\right]\right)(x,y)$

$= [\psi + c\phi](y,x)$

$= \psi(y,x) + (c\phi)(y,x)$

$= \psi(y,x) + c \cdot \phi(y,x)$

$= [A\psi](x,y) + c \cdot [A\phi](x,y)$

$= [A\psi](x,y) + [cA\phi](x,y)$

$= [A\psi + cA\phi](x,y)$

B) $\left[A^2\psi\right](x,y) = \left[A[A\psi]\right](x,y)$

$= [A\psi](y,x) = \psi(x,y)$

C) IF ψ IS AN EIGENFUNCTION OF A w/ E.V. λ, THEN:

$A^2\psi = A[\lambda\psi] = \lambda[A\psi] = \lambda^2\psi$

BUT $A^2 = \mathbb{1}$ SO $\lambda^2\psi = A^2\psi = \psi$

SO $\lambda^2 = 1 \Longrightarrow \lambda = \pm 1$

D) SINCE A HAS REAL (NO IMAGINARY COMPONENT] EIGENVALUES, A IS HERMITEAN.
(NOTE: THIS PROOF IS FLAWED. WHAT IS THE FLAW?)

A10 A) LET $\psi(x,y)$ BE AN ARBITRARY WAVE FUNCTION. THEN:

$$[H'\psi](x,y) = H'(x,y)\,\psi(x,y)$$

(i.e. H' IS "DIAGONAL IN THE SPATIAL REPRESENTATION")

$$\text{So } [[AH']\psi](x,y)$$

$$= [A[H'\psi]](x,y)$$

$$= (H'\psi)(y,x)$$

$$= H'(y,x)\,\psi(y,x)$$

SINCE H' IS SYMMETRIC

$$= H'(x,y)\,[A\psi](x,y)$$

$$= [H'[A\psi]](x,y)$$

$$= [[H'A]\psi](x,y)$$

KINETIC ENERGY

$$\overbrace{}$$

B) $H_0 = \frac{-\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + V(x,y)$

WHERE $V(x,y) = \begin{cases} 0 & \text{IF } \begin{matrix} 0 < x < a \\ \text{AND} \\ 0 < y < a \end{matrix} \\ \infty & \text{ELSE} \end{cases}$

AND HAS SAME REFLECTION SYMMETRY AND THUS COMMUTES.

FOR THE KINETIC ENERGY WE NOTE THAT:

$$\left(A \frac{\partial}{\partial x} \psi\right](x_0, y_0)$$

$$= \left(\frac{\partial}{\partial x} \psi\right)(y_0, x_0)$$

$$= \lim_{\Delta x \to 0} \frac{\psi(y_0 + \Delta x, x_0) - \psi(y_0, x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(A\psi)(x_0, y_0 + \Delta x) - (A\psi)(x_0, y_0)}{\Delta x}$$

$$= \left(\frac{\partial}{\partial y} A\psi\right](x_0, y_0)$$

LIKEWISE, $A\frac{\partial}{\partial y} = \frac{\partial}{\partial x} A$

SO $A\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial y} A \frac{\partial}{\partial x} = \frac{\partial^2}{\partial y^2} A$

AND $A\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} A$

SO $A\left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]$

$= \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) A$

SO $\left[A, H_0\right] = 0$