WEEK 4

SYMMETRIES AND EIGENSTATES

<u>INTRO</u>

IN PHYSICS WE EMPLOY SYMMETRY TO SIMPLIFY THE SOLUTIONS OF PROBLEMS THAT POSSESS IT. IN GENERAL, IF WE FIND A SYMMETRY OF A PROBLEM THEN WE SAVE ~~SOME~~ OURSELVES SOME WORK.

IN AN ABSTRACT SENSE, FINDING THE SYMMETRY IS JUST AS HARD AS DOING THE WORK WE SAVE BY FINDING THE SYMMETRY. IN A <u>PRACTICAL</u> SENSE, HOWEVER, WE HUMANS SEEM TO HAVE A NATURAL ABILITY OR INTUITION FOR FINDING SYMMETRIES. PHYSICISTS USE SYMMETRY PRINCIPLES BECAUSE THEY PLAY TO OUR STRENGTHS AS HUMANS.

<u>SECTION</u> 1 SY<u>MMETRIES</u>

~~WHILE~~ IN QUANTUM MECHANICS WE WILL DEFINE A SYMMETRY IN THE FOLLOWING WAY:

A SY<u>MMETRY</u> IS A HERMITEAN OPERATOR THAT CO<u>MMUTES</u> WITH THE HAMILTONIAN OF THE SYSTEM.

SO IF WE CALL THE SYMMETRY A AND THE HAMILTONIAN H, THEN A IS A SYMMETRY OF H IF:

$$[H, A] = 0$$

EASY

Q1 SHOW THAT IF A IS A SYMMETRY OF H, THEN IF $|\psi\rangle$ IS AN EIGEN-STATE OF H W/ EIGEN-VALUE λ, THEN $A|\psi\rangle$ IS <u>ALSO</u> AN EIGEN-STATE OF H W/ THE <u>SAME</u> EIGEN-VALUE λ

HINT: WHAT CAN YOU TELL ME ABOUT $HA|\psi\rangle$, KNOWING THAT $[H, A] = 0$?

MEDIUM

Q2 PROVE THE RELATED STATEMENT:

IF A IS A SYMMETRY OF H, THEN $\langle \psi' | A \psi \rangle = 0$, WHERE $|\psi\rangle$, $|\psi'\rangle$ ARE EIGEN-STATES OF H WITH EIGENVALUES λ, λ' RESPECTIVELY, WHERE $\lambda \neq \lambda'$.

HINT: SINCE H IS HERMITEAN, THEN

$$\langle H\psi' | A\psi \rangle = \langle \psi' | HA\psi \rangle$$

HARD

Q3 PROVE THIS IMPORTANT RESULT ABOUT SYMMETRIES:

IF A IS A SYM OF H,
THEN A & H ARE
"SIMULTANEOUSLY DIAGONALIZABLE",
I.E. THERE IS A BASIS
$$(|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_i\rangle, \ldots)$$
WHERE EACH BASIS VECTOR $|\psi_i\rangle$
IS ~~BO~~ AN EIGEN STATE OF BOTH
A & H.

HINT: TAKE ANY EIGEN BASIS OF
H:
$$(|\psi_{11}\rangle, |\psi_{12}\rangle, \ldots, |\psi_{ij}\rangle, \ldots)$$
WHERE $|\psi_{ij}\rangle$ IS THE j^{TH} EIGENSTATE OF H
THAT HAS AN EIGEN-VALUE OF λ_i.
HOW DOES A ACT ON THESE $|\psi_{ij}\rangle$?
CAN YOU COME UP WITH A PROCEDURE
FOR MAKING A NEW BASIS
$$(|\psi'_{11}\rangle, |\psi'_{12}\rangle, \ldots, |\psi'_{ij}\rangle, \ldots)$$
SO THAT $|\psi'_{ij}\rangle$ IS STILL AN EIGENSTATE
OF H BUT IS NOW ALSO AN EIGENSTATE
OF A?

DIGRESSION

SO WE HAVE SHOWN THAT IT IS ALWAYS POSSIBLE TO FIND A SET OF ENERGY EIGENSTATES WHERE EACH EIGENSTATE IS ALSO AN EIGENSTATE OF THE SYMMETRY OPERATOR. THESE STATES ARE SAID TO HAVE "DEFINITE SYMMETRY" AND THEIR EIGENVALUE WITH RESPECT TO THE SYMMETRY IS TERMED THE "SYMMETRY" OF THE EIGENSTATE.

NOTE, HOWEVER, THAT WE CAN MAKE EIGENSTATES WITH NO DEFINITE SYMMETRY IF, FOR INSTANCE, WE ADD TWO ~~EIGE~~ DEGENERATE EIGENSTATES (SAME EIGENVALUE W.R.T. H) ~~B~~ WITH DIFFERENT SYMMETRY (DIFFERENT EIGENVALUES W.R.T. THE SYMMETRY OPERATOR A).

IN PRACTICE SUCH DEGENERACIES ARE CONSIDERED COINCIDENTAL AND ARE KNOWN AS "ACCIDENTAL DEGENERACIES". UNDER NORMAL CIRCUMSTANCES ANY PARTICULAR ENERGY LEVEL IS MADE UP OF EIGENSTATES OF IDENTICAL SYMMETRY.

A SEEMING EXCEPTION IS THE NON-RELATIVISTIC HYDROGEN ATOM. HERE THE SYMMETRY OPERATOR WOULD APPEAR

TO BE L^2, ASSOCIATED WITH THE SPHERICAL SYMMETRY OF THE COULOMB POTENTIAL. IN THIS CASE THERE WOULD APPEAR TO BE AN ACCIDENTAL DEGENERACY SINCE THERE ARE STATES OF DIFFERENT SYMMETRY (DIFFERENT l QUANTUM NUMBER) BUT EQUAL ENERGY (SAME n QUANTUM NUMBER).

IT TURNS OUT THAT THE HYDROGEN ATOM ACTUALLY POSSESSES A MORE EXOTIC SYMMETRY, ASSOCIATED WITH THE FACT THAT THE COULOMB POTENTIAL HAS CLOSED CLASSICAL ORBITS (KEPLERIAN ELLIPSES). ALL STATES OF EQUAL n QUANTUM NUMBER TURN OUT TO BE STATES OF IDENTICAL SYMMETRY WITH RESPECT TO THIS MORE EXOTIC SYMMETRY, SO IN FACT THERE IS NO ACCIDENTAL DEGENERACY.

EASY

Q4 SHOW THAT IF A IS A SYMMETRY OF H AND THERE IS ONLY ONE EIGENBASIS OF A, THEN THIS BASIS ALSO DIAGONALIZES H.

HINT: APPLY RESULT OF Q3

SECTION 2 — SYMMETRIES AND DEGENERATE P.T.

THE RECIPE FOR DEGENERATE PERTURBATION THEORY CAN BE STATED CONCISELY AS:

"FOR EVERY DEGENERATE SUBSPACE OF THE UNPERTURBED HAMILTONIAN, DIAGONALIZE THE RESTRICTION OF THE PERTURBATION TO THAT DEGENERATE SUBSPACE"

OK THAT'S A MOUTHFULL. LET'S BREAK IT DOWN.

HERE IS THE DEFINITION OF A DEGENERATE SUBSPACE:

"THE DEGENERATE SUBSPACE OF EIGENVALUE λ WITH RESPECT TO THE OPERATOR H IS THE SET OF ALL VECTORS THAT ARE EIGENVECTORS OF H WITH EIGEN-VALUE λ"

WHEN IT IS CLEAR WHICH OPERATOR IT IS WITH RESPECT TO, WE WILL DENOTE THE DEGENERATE SUBSPACE OF EIGENVALUE λ AS Λ (UPPERCASE λ).

EASY

Q5 SHOW THAT A DEGENERATE SUBSPACE, AS DEFINED ABOVE, IS ACTUALLY A SUBSPACE.

HINT: A SUBSET $S \subset V$ OF A VECTOR SPACE V IS A SUBSPACE IF IT IS CLOSED UNDER ADDITION AND SCALAR MULTIPLICATION. i.e. A SUBSET S IS A SUBSPACE IF FOR ANY TWO VECTORS $|\psi\rangle$ AND $|\phi\rangle$ IN S AND ANY SCALAR (COMPLEX NUMBER) c, IT HOLDS THAT:

$$|\psi\rangle + c|\phi\rangle \text{ is in } S \text{ as well}$$

THE RESTRICTION OF AN OPERATOR IS DEFINED AS FOLLOWS:

THE RESTRICTION OF AN OPERATOR B THAT ACTS ON VECTORS IN V TO A SUBSPACE S OF V IS DENOTED $B|_S$ AND ACTS ON VECTORS IN S IN THE FOLLOWING WAY:

$$B|_S v = P_S B i_S v, \quad v \in S$$

WHERE i_S IS CALLED THE "INCLUSION MAPPING" FOR S AND SIMPLY DOES NOTHING. i_S TAKES VECTORS FROM S ~~AND~~ TO V ACCORDING TO THE SIMPLE RULE:

$$i_S v = v, \quad v \in S$$

IT IS INCLUDED SIMPLY TO GET THE DOMAINS AND RANGES TO WORK OUT.

THE "PROJECTION OPERATOR" P_S OF A SUBSPACE S TAKES VECTORS ON V ~~TO~~ AND SENDS THEM TO S BY CHOPPING OF THE COMPONENT OF v ~~THE VECTOR~~ V THAT IS PERPENDICULAR TO S:

$$P_S v = P_S (v_{\parallel} + v_{\perp}) = v_{\parallel}$$

WHERE $v_{\parallel}$ IS IN S AND $v_{\perp}$ IS PERPENDICULAR TO S, i.e.:

$\langle w | v_{\perp} \rangle = 0$ FOR ANY $|w\rangle$ IN S

NOTE: THE DECOMPOSITION OF A VECTOR v INTO PARALLEL AND PERPENDICULAR COMPONENTS $v_{\parallel}$ AND $v_{\perp}$, i.e.:

$$v = v_{\parallel} + v_{\perp}$$

IS <u>UNIQUE</u>. THERE IS ONLY ONE WAY TO DO IT.

~~Q6 SHOW THAT FOR ANY SUBSPACE S THAT $P_S^2 = P_S P_S = P_S$ (THE IDENTITY OPERATOR) AND THAT THIS IMPLIES THAT P_S HAS ONLY TWO EIGENVALUES~~

SO WHEN WE PUT THE DEFINITION BACK TOGETHER, WE FIND THAT, INDEED AS WE CLAIMED, AN OPERATOR B RESTRICTED TO A SUBSPACE S TAKES VECTORS IN S AND MAPS THEM TO VECTORS ALSO IN S:

$$B|_S = P_S B i_S$$

" i_S TAKES VECTORS IN S AND SENDS THEM TO V (THE WHOLE VECTOR SPACE) "

OR IN OTHER WORDS:
(MATH NOTATION)

$$i_S : S \longrightarrow V$$

" B TAKES VECTORS FROM V TO V "

$$B : V \longrightarrow V$$

" P_S TAKES VECTORS FROM V TO S "

$$P_S : V \longrightarrow S$$

SO THAT:

$$B|_S : S \xrightarrow{i_S} V \xrightarrow{B} V \xrightarrow{P_S} S$$

$$S \longrightarrow S$$

NOW CONSIDER AN OPERATOR A THAT IS A SYMMETRY OF BOTH H_0 AND H'.

MEDIUM

Q6 SHOW THAT A "COMMUTES" WITH i_Λ & P_Λ, WHERE Λ IS THE DEGENERATE SUBSPACE OF EIGENVALUE λ W.R.T. H_0, IN THE FOLLOWING SENSE:

$$A\, i_\Lambda = i_\Lambda\, A|_\Lambda$$

&

$$P_\Lambda A = A|_\Lambda P_\Lambda$$

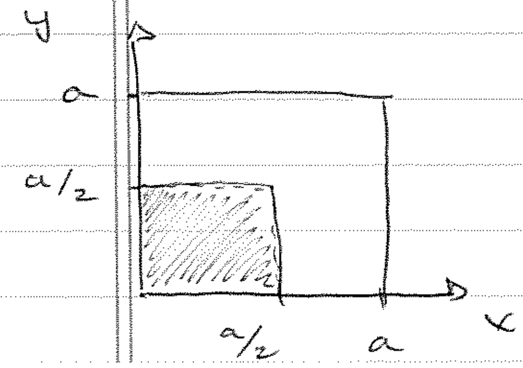
EASY

Q7 USE THE RESULTS FROM Q6 TO SHOW THAT $A|_\Lambda$ IS A SYMMETRY OF $H'|_\Lambda$, OR ~~IAG~~ IN OTHER WORDS THAT:

$$\left[A|_\Lambda , H'|_\Lambda\right] = 0$$

SECTION 3 — 2D INFINITE SQUARE WELL

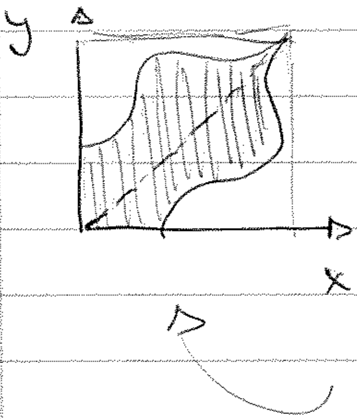
CONSIDER THE FOLLOWING PERTURBATION ON THE 2D ∞ S.W. THAT YOU SAW IN CLASS:



$$H'(x,y) = \begin{cases} V_0 & \text{IF } 0 < x < a/2 \text{ AND } 0 < y < a/2 \\ 0 & \text{OTHERWISE} \end{cases}$$

MEDIUM

Q8 FROM VISUAL INSPECTION (OR "INTUITION"), CAN YOU IDENTIFY QUALITATIVELY THE GEOMETRIC SYMMETRY OF THE PERTURBATION?



HINT: THE PERTURBATION DEPICTED ON THE LEFT HAS THE SAME SYMMETRY.

~~OR~~ WE CAN ASSOCIATE A QUANTUM MECHANICAL SYMMETRY OPERATOR WITH THE GEOMETRIC SYMMETRY WE JUST FOUND IN THE FOLLOWING WAY:

"THE SYMMETRY OPERATOR ACTS ON WAVE FUNCTIONS $\psi(x,y)$ TO PRODUCE A WAVE FUNCTION WHOSE VALUE AT SOME POINT x, y IS THE VALUE OF THE ORIGINAL FUNCTION AT THE TRANSFORMED POINT."

SINCE OUR REFLECTION GEOMETRICALLY SENDS A POINT (x, y) TO THE POINT (y, x) (i.e. IT SWITCHES THE COORDINATES), THIS MEANS THE SYMMETRY OPERATOR A ASSOCIATED WITH OUR REFLECTION SYMMETRY ACTS ON WAVE FUNCTIONS IN THE

FOLLOWING WAY:

$$[A\psi](x,y) = \psi(y,x)$$

~~MEDIUM~~

Q9 SHOW THAT A AS DEFINED DIRECTLY ABOVE HAS THE FOLLOWING ~~props~~ PROPERTIES:

EASY A) A IS A LINEAR OPERATOR, I.E. FOR ANY TWO WAVE FUNCTIONS $\psi(x,y)$ & $\phi(x,y)$ AND SCALAR c WE HAVE:

$$\left[A[\psi + c\phi]\right](x,y)$$

$$= [A\psi](x,y) + c\,[A\phi](x,y)$$

EASY B) ~~A is Hermitian~~ $A^2 = 1$

MEDIUM C) USE B) TO SHOW THAT A HAS ONLY TWO EIGENVALUES, 1 & -1

EASY D) A IS HERMITIAN

HINT: c) ASSUME ψ IS AN EIGENFUNCTION OF A AND APPLY A TWICE. SHOW THAT THE EIGENVALUE λ OF ψ HAS THE PROPERTY $\lambda^2 = 1$

d) WHAT IS TRUE ABOUT THE EIGENVALUES OF HERMITIAN OPERATORS?

EASY

Q10 A) SHOW THAT A IS A SYMMETRY OF H'

MEDIUM B) SHOW THAT A IS A SYMMETRY OF H_0

HINT: A) IT IS ~~ENOUGH~~ ENOUGH TO SHOW THAT

$$H'(x,y) = H'(y,x)$$

B) THE POTENTIAL ENERGY TERM IN H_0 COMMUTES WITH A FOR THE SAME REASON AS H':

$$V(x,y) = V(y,x).$$

FOR THE KINETIC ENERGY TERM, FIRST SHOW THAT:

$$A \frac{\partial}{\partial x} \psi = \frac{\partial}{\partial y} A\psi$$

AND (LIKEWISE):

$$A \frac{\partial}{\partial y} \psi = \frac{\partial}{\partial x} A\psi$$

CONSIDER THE 2^{ND} LOWEST ENERGY LEVEL FOR THE UNPERTURBED HAMILTONIAN H_0.

IT IS DEGENERATE SINCE:

$$\psi_A(x,y) \equiv \psi_1^\circ(x)\,\psi_2^\circ(y)$$

AND

$$\psi_B(x,y) \equiv \psi_2^\circ(x)\,\psi_1^\circ(y)$$

($\psi_i^\circ(x,y)$ IS THE i^{TH} EIGENFUNCTION OF THE 1D ∞ S.W.)

HAVE THE SAME ENERGY.

Q11 LET Λ BE THE DEGENERATE SUBSPACE SPANNED BY ψ_A + ψ_B (DEFINED ABOVE). FIND THE "CORRECT" ZEROTH ORDER WAVEFUNCTIONS FOR Λ BY DOING THE FOLLOWING:

A) WHAT IS $A\psi_A$? ~~$A\psi_B$~~
$A\psi_B$?

B) CONSTRUCT THE MATRIX FOR $A|_\Lambda$ IN THE $(|\psi_A\rangle, |\psi_B\rangle)$ BASIS:

$$\begin{bmatrix} \langle \psi_A | A|_\Lambda |\psi_A\rangle & \langle \psi_A | A|_\Lambda \psi_B\rangle \\ \\ \langle \psi_B | A|_\Lambda \psi_A\rangle & \langle \psi_B | A|_\Lambda \psi_B\rangle \end{bmatrix}$$

HINT: REMEMBER THAT IF $|\psi\rangle \in \Lambda$ THEN $A|\psi\rangle \in \Lambda$ AS WELL, SO

$$A|_\Lambda |\psi\rangle = A|\psi\rangle$$

C) SHOW THAT THERE IS ONLY ONE EIGENBASIS FOR $A|_\Lambda$ BY SHOWING THAT $A|_\Lambda$ HAS TWO DIFFERENT EIGENVALUES.

HINT: IF A 2×2 MATRIX HAS ~~ALL THE SAME EIGENVALUES~~ BOTH EIGENVALUES EQUAL, THEN IT HAS WHAT FORM? i.e. IT MUST BE EQUAL TO A CONSTANT TIMES WHAT SPECIAL MATRIX WHOSE NAME ~~THAT~~ STARTS WITH "i"?

D) WHAT MUST THESE EIGENVALUES BE?

HINT: Q9 c)

E) FIND THE EIGENVECTORS OF $A|_\Lambda$

~~Hint:~~ IN TERMS OF $|\psi_A\rangle$ AND $|\psi_B\rangle$

HINT: $A\left[\alpha|\psi_A\rangle + \beta|\psi_B\rangle\right] = \pm\alpha|\psi_A\rangle$

FOR WHICH VALUES α, β?

F) ARE THESE EIGENVECTORS ALSO EIGENVECTORS OF $H'|_\Lambda$?

HINT: Q4

Q12 WHAT ARE THE FIRST ORDER ENERGY CORRECTIONS TO THE STATES IN Λ?